# Linear systems

**Def** A linear system is consistent if it has a solution.
A linear system is inconsistent if it doesn't have a sol.

How to solve it?

Use matrix & row echelon form. (for the first non-zero entry of each row, every entry is 0 below the non-zero entry)

e.g.

$$x + y + z = 5$$
$$x + 2y - z = 9$$
$$3x + 5y - z = 23$$

coefficents of the system (called 'augmented matrix')

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & -1 & 9 \\ 3 & 5 & -1 & 23 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 2 & -4 & 8 \end{array}\right) \begin{array}{l} \\ ② - ① \\ ③ - 3 \cdot ① \end{array}$$

pivots

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ ③ - 2 \cdot ② \end{array}$$

free variable

$$x + y + z = 5 \qquad x = 5 - (4 + 2t) - t = 1 - 3t$$
$$y - 2z = 4 \qquad y = 4 + 2t$$
$$z = t$$

↳ set $z$ as $t$ since it is free

pivot: first non-zero entry of each row

free variable: if i-th column is a non-pivot column

then we say $x_i$ is a free variable.

$$a_{11}x_1 + \cdots + a_{1m}x_m = c_1$$
$$a_{21}x_1 + \cdots + a_{2m}x_m = c_2$$
$$\vdots$$
$$a_{n1}x_1 + \cdots + a_{nm}x_m = c_n$$

If $c_1 = \cdots = c_n = 0$, we say the system is homogeneous.

Prop Consider a row echelon form of a homogeneous system (not augmented). If

1) # of pivot = # of column: exactly one solution $\mapsto \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

2) # of pivot < # of column: infinitely many solutions

(more precisely, # of free variables (# of col − # of pivots) is the # of indep sols)

## Vector spaces

Def A vector space V is a set paired w/ a scalar field $\mathbb{F}$ and equipped w/ binary operations $+$ and $\cdot$ such that

- $+ : V \times V \to V,$

  $(x, y) \mapsto x+y$

- $\cdot : \mathbb{F} \times V \to V$

  $(c, x) \mapsto c \cdot x$

where $+$ and $\cdot$ are compatible (distribution & associativity)

and $(V, +)$ is an additive group w/ identity $0$

- Usually, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{Z}_2$

e.g $\mathbb{R}^n, \mathbb{C}^n$

**Def** A subset of a vector space is a subspace if it contains $0$ and is closed under addition and scalar multiplication

- A subspace is a vector space contained in some larger vector space.

How to make it?

**Def** Let $V$ be a vector space and $S$ a subset of $V$. The span of $S$ ($\mathrm{Span}(S)$) is the smallest subspace of $V$ containing $S$.

- $\text{Span}(S)$ = the smallest subspace of $V$ containing $S$
  = the intersection of all subspaces containing $S$
  = the set of all linear combinations of $S$.

  ↳ $c_1v_1 + \cdots + c_nv_n$

  $c_i \in \mathbb{F}$, $v_i \in S$

Def A set of vectors $\{v_1, \cdots, v_n\}$ is linearly independent if for $c_1, \cdots, c_n \in \mathbb{F}$

$$c_1v_1 + \cdots + c_nv_n = 0 \Rightarrow c_1 = \cdots = c_n = 0$$

Def A basis of a vector space $V$ is a linearly independent subset that spans $V$.

e.g. $\mathbb{R}^2$, $\{e_1 = \binom{1}{0}, e_2 = \binom{0}{1}\}$

$\{e_1 + e_2, e_1 - e_2\}$

$\vdots$

Thm Every vector space has a basis.

**Thm** Two bases of a vector space have the same cardinality

**Def** The dimension of a vector space is the number of elements in a basis.

**e.g** $\dim(\mathbb{R}^n) = n$ $\quad \dim(\mathbb{C}^n) = \begin{cases} 2n & (\text{as } \mathbb{R}\text{-v.s.}) \\ n & (\text{as } \mathbb{C}\text{-v.s}) \end{cases}$

## Linear transformations

**Def** Let $V, W$ be vector spaces over $\mathbb{F}$.
A linear transformation $T : V \to W$ is a map s.t.

$$T(cu+v) = cT(u) + T(v) \qquad c \in \mathbb{F},\ u,v \in V.$$

**Def** Let $T: V \to W$ be a linear transformation.
The null space of $T$ is

$$N(T) = \{ v \in V : T(v) = 0 \}$$

The range of $T$ is

$$R(T) = \{ T(v) : v \in V \}$$

$N(T)$ is a subspace of $V$
$R(T)$ is a subspace of $W$.

Def Two vector spaces $V$ and $W$ are isomorphic if there is an invertible linear map.
i.e. $T : V \to W$

$$T^{-1} \circ T = Id_V \qquad T \circ T^{-1} = Id_W$$

We write $V \cong W$

Thm (Classification of vector spaces) Let $V$ be a vector space over $\mathbb{F}$. If $\dim(V) = n$, then

$$V \cong \mathbb{F}^n$$

Thm (Matrix representation) $T : \mathbb{R}^m \to \mathbb{R}^n$ a linear transformation, Then there exist a matrix $A \in \mathbb{R}^{n \times m}$ s.t

$$T(v) = Av \qquad v \in \mathbb{R}^m$$

- the i-th Column of $A$ is $T(e_i)$

**Def** $\operatorname{rank}(T) := \dim(R(T))$

$\operatorname{rank}(A) :=$ # of indep columns of $A$

(= # of pivots of an echelon form)

$\operatorname{null}(T) := \dim(N(T))$

$\operatorname{null}(A) :=$ # of indep sols of $Ax = 0$

**Thm** $T: \underset{\dim = m}{V} \to \underset{\dim = n}{W}$ $\qquad A \in \mathbb{R}^{n \times m}$

$\operatorname{rank}(T) + \operatorname{null}(T) = m$

$\operatorname{rank}(A) + \operatorname{null}(A) = m$